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SUBSONIC, TRANSONIC, SUPERSONIC NOZZLE FLOW FIELDS

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ABSTRACT

The present article gives the general form of the equation for the two-dimensional constant steady irrotational isentropic flow of an ideal gas in conformal curvilinear coordinates. Simplified equation for the stream function and its general solution are given using the streamlines and equipotential lines of the corresponding incompressible potential flow as coordinates. The results are applied to nozzle flow, giving the solution for nozzle flow from subsonic through transonic to the supersonic range, for any desired choice of radius of curvature of throat wall, contraction ratio and largest angle of inclination of the wall. This solution is valid *for different ratios of specific heats*.

As an example of the application, we have calculated the flow characteristics of a typical nozzle. These include: line of constant Mach number for subsonic, transonic and supersonic nozzle flow; velocity line, line of constant Mach number, influence line, limiting characteristic line, branchline and equitemporal lines for supersonic nozzle flow.

This method can be extended to flow around an obstacle, over-weir flow in the nozzle and flow through a grating. Particularly, one can obtain better results in the transonic region. Furthermore, the method can be extended to cases where there are chemical reactions under equilibrium or non-equilibrium conditions, *or where there exists axial symmetry*.

SYMBOLS

A_k function, Equation (14)

a   velocity of sound, Equation (3)

B_{lm} (ξ,α) function, Equations (33), (34)

b   half-height of nozzle, throat, Equation (1)

D   determinant, Equation (9)

E_k (ξ,η) function relating to the transformations, Equation (14)

$f(\zeta)$ transformation of conformal curvilinear coordinates, Equation (6)

$G_1(\xi,\eta)$ function, Equation (33)

$g(\xi)$ flow-related function, Equation (12)

$g_s(0)$ flow-related function for maximum flow at the throat, Equations (18), (22)

H (ξ,η) amplification factor in going from z-plane to ζ-plane, Equation (6)

l point of inflection at the wall, Figure 1

$i = \sqrt{-1}$

$k = \psi_b/\psi_s$ nozzle flow factor, Equation (25)

M Mach number, Equation (4)

N_m (ξ,η,α) functions, Equations (33), (34)

n nozzle contraction ratio Equations (31), (32)

p pressure, Equation (3)

q velocity, Equation (1)

r radius of curvature of throat wall, Equations (31), (32)

s distance along flow line, Equation (27)

T absolute temperature of gas, Equation (2)

u velocity component along ξ direction, Equation (2)

v velocity component along η direction, Equation (2)

x rectangular coordinate, Equation (6)

y rectangular coordinate, Equation (6)

z = x + iy complex variable in rectangular coordinates, Equation (6)

$\alpha = ch^{-1}$ $(\sigma ch\lambda)$, Equation (33)

$\beta = -i\alpha$, Equation (37)

γ ratio of specific heats, Equation (3)

$\delta = -tg^{-1}(\psi_\xi/\psi_\eta)$ angle between flow and η direction, Equation (26)

$\zeta = \xi + i\eta$ complex variable in conformal curvilinear coordinates, Equation (6)

η conformal curvilinear coordinate, Equation (6)

η_b nozzle shape factor, Equation (28)

θ_1 largest angle of inclination of nozzle wall, Equations (31), (32)

λ nozzle shape factor, Equation (28)

$\mu = tg^{-1}(M^2-1)^{-1/2}$ Mach angle, Equation (26)

ν positive whole number, Equation (14)

ξ conformal curvilinear coordinate, Equation (6)

ξ_1 point of inflection at wall, Equations (31), (32)

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ρ density of gas, Equation (3)

σ nozzle shape factor, Equation (28)

τ time spent in nozzle by gas particles, Equation (27)

ϕ potential function, Equation (5)

ψ stream function, Equation (5)

ψ_s largest flow in nozzle, Equations (19), (23)

$w(\xi)$ function, Equation (25)

Subscripts:

* critical parameter for which stream velocity equals local velocity of sound

b nozzle wall parameter

j ordinal number

k ordinal number

l ordinal number

m ordinal number

x partial derivative w.r.t. x

y partial derivative w.r.t. y

η partial derivative w.r.t. η

ξ partial derivative w.r.t. ξ

I. INTRODUCTION

Nozzles have wide applications in engineering. As early as the 80's of the 19th Century, Laval [1] used convergent-divergent nozzles in steam turbines to obtain supersonic flows. During the same period of time, Reynolds [2] analyzed theoreitcally one-dimensional supersonic nozzle flows. Early in this century Meyer [3] was the first to analyze two-dimensional supersonic nozzle flows. In the 30's, Taylor [4], Hocker [5] and Göertler [6] separately studied the higher subsonic nozzle flow near the thorat wall that is partially supersonic. After the 40's, the development of supersonic aircraft and jet propulsion technology spurred research on supersonic flows, resulting in a large quantity of work on theoretical analysis and designing. From the viewpoint of applications, when the ratio of the radius of curvature of throat wall to the half-height of the throat passage is very large, the design problem of the supersonic nozzle is basically solved. In the past decade, high acceleration nozzles became necessary to shorten the nozzle length of the generator of rockets and to in-crease the efficiency with which the gas-propelled, excited light nozzle freezes high temperature gases under non-equilibrium condi-tions. In order to achieve this, the radius of curvature of the throat wall needs to be decreased and the angles of convergence and divergence have to be increased. The presently available anal-yses and design methods for supersonic nozzle flows do not meet this design requirement and hence, are not applicable to the study of supersonic flows in nozzles with small radius of curvature of throat wall.

A large quantity of information has been accumulated regard-ing the study of supersonic nozzle flows. According to methods of treating the problem, one can divide it into eight classes: expansion in power series [3-6,8,15,51], hodograph method [7,9,10, 19,23,25,27,32-34,37,44,46-48,50,52-58,60], iterative substitution relaxation method [13,14], inversion method [16,43,67-69,80,87],

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expansion in small parameters [59,64,65,70,93], band interration [61,62,63], time correlation method [71,72,77,79,81,84,85] and method of imaginary characteristic line Prandtl's relation [82].

In the method of expansion in power series, the velocity potential or velocity component is expressed as a power series in the position coordinates. A definite velocity distribution is assumed and an approximate solution for the nozzle flow is obtained by finding the expansion coefficients. For example, Oswatitsch et al.[8] assume that all the equivelocity lines are quadratic curves in positive coordinates. Sauer [15] assumes that the velocity distribution on the central streamline at the throat bears a linear relation to the abscissa. This method gives an approximate solution for the case of larger radius of curvature of throat wall.

In the hodograph method, the velocity components are used as coordinates, and the nozzle flow properties are studied on the hodograph plane. Accordingly, the equations become linear, but the boundary conditions become more complicated [91]. There are many papers in which the nozzle flow is treated using the hodograph method. Ringleb [7] was the first to use this method to obtain the singular solution for flow around a semi-infinite plane wall that contains partial supersonic regions. Temple et al.[9] elucidated this flow and pointed out that it is equivalent to the nozzle flow near the throat wall that is partially supersonic. Frankl [19] and Lighthill [27] independently introduced the concent of branchline of transonic nozzle flow in the hodograph plane and used methods of approximation to discuss the flow characterics in the vicinity of the velocity point on the center line. Tomotika et al. [32] used an assumed gas to analyze changes in subsonic and supersonic nozzle flow. This has definite significance in the understanding of flow characteristics in the throat region. Cherry [34,37,52,56] used the method of iterative addition of potentials to obtain an accurate solution for the supersonic flow in nozzles

with particular wall contours. This study is important in contributing to the understanding of the flow in the transonic region of the nozzle. Because the boundary conditions are difficult to control, it cannot be conveniently applied in engineering designs.

The iterative substitution relaxation method is based on the relaxation method. The supersonic nozzle flow is discussed using incompressible potential flow and equipotential line as coordinates. Green et al.[13] gave the basic equations for this method and Fox et al.[14] applied the method to an example of supersonic nozzle flow problems. Because of the limitations on the choice of the form of the solution and the boundary conditions, no practically useful results were obtained.

The inversion method was first developed by Friedrichs [16] who also treated the problem of nozzle flow. Liepman [43] used this method to treat two-dimensional nozzle flow; Hopkins et al. [67,69] and Pirumov et al [68] separately treated axially symmetrical nozzle flow. This method converges rather slowly and the boundary conditions cannot be effectively controlled. Brodsky [78] adopted the method of finite differences to solve the equations of the inversion method and discussed supersonic nozzle flow. He found that oscillations of the solution occurred in the subsonic region. The reason was that the boundary conditions for elliptical equations could not be determined from the velocity distribution on the center line alone.

Expansion in small parameters was first developed by Hall [59] who used the reciprocal of the radius of curvature of the throat wall as the small parameter and expressed the geometric boundary of the nozzle and the velocity components in terms of power series in this small parameter. Assuming a given relation between velocity and the abscissa, he obtained the first three orders of the solution. The first order solution was obtained under the

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assumption of a linear relationship between the velocity and the abscissa. The result agrees with Sauer's [15]. The second and third order solutions are improved over those obtained by Sauer and Hall [15]. Similarly, Kliegel et al.[70] used the reciprocal of the radius of curvature of the throat wall as the small parameter. He found an approximate solution to the entire nozzle flow by including in the one-dimensional flow a correction in the small parameter. The above results are applicable to the case of large radius of curvature of throat wall. Kliegel et al.[73] used double cylindrical coordinates and took the reciprocal of radius of curvature of throat wall increased by 1 as the small parameter in the expansion of the velocity components. By comparing with Hall's [59] series, the coefficients of expansion were determined. This improved the convergence of Hall's series.

Band integration is a method that employs band integration of supersonic flow around a blunt-headed body to calculate nozzle flow. Alikhashchkin et al.[61] used one and two bands separately to compute the flow characteristics of a two-dimensional nozzle with given shapes of the wall. Holt [62] used one band to compute the surface contour lines of a nozzle with given velocity distribution on the center line. Favorskiy [63] used two bands to compute the flow characteristics of an axially symmetric nozzle with given wall contours. In this method, each band contains a "saddle point" as a singular point which places a limit on the available number of bands in practical applications. Besides, the method integration only gives the average value between the bands and cannot be used to characterize local flow. In general, this method is not suitable for nozzles with small radius of curvature.

In the time correlation method for the solution of transonic flow, the mixed equations are reorganized into hyperbolic equations by introducing the time variable. The method of differences is used to obtain a solution for unstable motion of the gas described by the hyperbolic equations. In the limit of infinite time,

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an approximate solution for stable flow is obtained. This method offers a new approach to the solution of the transonic flow problem. This method is one of the most effective for non-isentropic flows, i.e., when there are excited waves. The major works using this method are by Migdal et al.[71], Ivanov et al.[72], Wehofer et al.[77], Laval [74], Serra [81], Brown et al.[84], Cline et al. [85]. Because the time variable is introduced to convert a constant, stable flow problem into one with unstable flow, the problem is complicated by the additional independent variable.

The method of imaginary characteristic line Prandtl's relation came about as follows. Armitage [66] and Rao et al.[74] used the method of imaginary characteristic line to compute the flow in the subsonic section. Extension to the sound velocity line was made and Prandtl's relation was used to treat the so-called mixed region between the sound velocity line and the limiting characteristic line [82]. Writing the series solution for the mixed region in the form of Prandtl's relation brings about convergence only in the form of the solution, the actual solution of the subsonic flow region remains complicated. Furthermore, there is also the problem of matching the solutions in the two regions.

In summary, there are many methods for studying nozzle flow. However, the properties of the mathematical equations for the transonic flow region at the throat presented great difficulties in the study of nozzle flow and some of the important problems of nozzle flow do not have a satisfactory solution. For instance, the relationship between the flow characteristics in the transonic region at the throat and the shape of the wall are difficult to determine, especially for very small radius of curvature of the throat, in which case the flow in the throat region is very nonuniform. Moreover, the lack of a simple method to solve the problem of transonic flow makes it very difficult to determine the initial conditions in the design of the divergent section of the nozzle.

Owing to the above reasons, empirical or semi-empirical methods have to be used in the engineering design of nozzles at present. These include the following: The method of Busemann-Atkin [16,28,31,88,92] converts radial flow into parallel flow. Because of its dependence on design experience and skills in the determination of throat wall contour, this method has considerable variability. Puckett et al.[24,29,30,83,86] used the assumption of linear velocity line in specifying the initial conditions for the design of the divergent section. This assumption applies approximately in the case of large radius of curvature of throat. For the case of small radius of curvature of throat, where the velocity line is far from being linear [75,76] this assumption is not reasonable [82,83]. Others have used the approximate results obtained from the above theoretical analyses as initial conditions in the design of the divergent section. Examples are found in the series method of Sauer [15], expansion in small parameters by Hall [59], and other semi-empirical methods [38,41]. In the design of the supersonic section, Guderly-Rao et al.[40,42,45,49] designed large-thrust nozzles with given initial conditions according to the principles of component transformation. At present, the design methods used in engineering apply only to nozzles with large radius of curvature of the throat wall. The problem of initial conditions for the case of small radius of curvature is what needs to be solved urgently.

With respect to the above problem, we have given the general form of the equation for the two-dimensional constant steady irrotational isentropic flow of an ideal gas in conformal curvilinear coordinates. Simplified equation for the stream function and its general solution are given using the streamlines and equipotential lines of the corresponding incompressible potential flow as coordinates. The results are applied to nozzle flow, giving the solution for nozzle flow from subsonic through transonic to the supersonic range with any desired choice of radius of curvature of throat

wall, contraction ratio and largest angle of inclination of the
wall. As an example, we have calculated the flow characteristics
of a typical nozzle.

II. FLOW EQUATIONS IN CONFORMAL CURVILINEAR COORDINATES

Consider the two-dimensional constant steady irrotational
isentropic flow of an ideal gas. Dimensionless quantities are
introduced according to the symbols listed in the list of symbols.
These are denoted with a "-" above the corresponding symbol:

$$\left.\begin{array}{llll} \bar{x} = x/b, & \bar{y} = y/b, & \bar{z} = \bar{x} + i\bar{y}, & \bar{r} = r/b \\ \bar{u} = u/a_*, & \bar{v} = v/a_*, & \bar{q} = (\bar{u}^2 + \bar{v}^2)^{\frac{1}{2}} \\ \bar{a} = a/a_*, & \bar{p} = p/p_*, & \bar{\rho} = \rho/\rho_*, & \bar{T} = T/T_* \\ \bar{\tau} = \tau a_*/b, & \bar{\Phi} = \Phi/a_* b, & \bar{\psi} = \psi/a_* \rho_* b, & M = \bar{q}/\bar{a} \end{array}\right\} \qquad (1)$$

In this paper, only dimensionless quantities will be used.
Hence, the "-" will be omitted.

The conservation of mass, momentum and energy of an ideal gas
in two-dimensional constant steady irrotational isentropic flow
can be represented in arbitrary orthogonal curvilinear coordinates
(ξ, η), respectively, as:

$$\left.\begin{array}{l} (H_2 \rho u)_\xi + (H_1 \rho v)_\eta = 0 \\ (H_1 u)_\eta - (H_2 v)_\xi = 0 \\ T = \dfrac{\gamma+1}{2} - \dfrac{\gamma-1}{2} q^2 \end{array}\right\} \qquad (2)$$

in which $H_1 = (x_\xi^2 + y_\xi^2)^{\frac{1}{2}}$, $H_2 = (x_\eta^2 + y_\eta^2)^{\frac{1}{2}}$ are the reciprocals of
Lamé coefficients in the orthogonal curvilinear coordinates (ξ, η)
[89].

The state equation, isentropic relation and the relation be-
tween sound velocity a and temperature T of an ideal gas are given
respectively by:

$$p = \rho T, \quad p = \rho^\gamma, \quad a = T^{\frac{1}{2}} \qquad (3)$$

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From Equations (2) and (3), one obtains the relations among
density ρ, Mach number M and velocity q to be:

$$\rho = \left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2} q^2\right)^{\frac{1}{\gamma-1}}$$
$$M = q \left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2} q^2\right)^{-\frac{1}{2}}$$

(4)

From the equation for conservation of mass, Equation $(2a)$[1]
and that for conservation of momentum, i.e., irrotationality,
Equation (2b), we define stream function ψ and potential function
ϕ to be respectively:

$$\psi_\xi = -\rho H_1 v, \quad \psi_\eta = \rho H_2 u$$
$$\Phi_\xi = H_1 u, \quad \Phi_\eta = H_2 v$$

(5)

When $H_1 = H_2$, i.e., when the transformation from orthogonal
linear coordinates x,y to curvilinear coordinates ξ,η is conformal,
according to the theory of complex variable [90], the following
relation exists between coordinates, x,y and ξ,η:

$$z = f(\zeta), \quad z = x + iy, \quad \zeta = \xi + i\eta$$
$$H_1 = H_2 = H = \left|\frac{dz}{d\zeta}\right|$$

(6)

where H is the amplification factor of z plane with respect to
[w. r. t.] ζ plane. In conformal curvilinear coordinates, from
equations (4a) and (5), one obtains the following relation between
density ρ, velocities u, v and stream function ψ.

$$\rho^{\gamma+1} - \frac{\gamma+1}{2} \rho^2 + \frac{\gamma-1}{2H^2} (\psi_\xi^2 + \psi_\eta^2) = 0$$
$$u = \psi_\eta/\rho H, \quad v = -\psi_\xi/\rho H$$

(7)

Using $H_1 = H_2 = H$ and Equation (4a), one can eliminate u,v, ϕ_ξ,
ϕ_η and ρ_ξ, ρ_η from Equation (5) to obtain the equation for stream
function in conformal curvilinear coordinates

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[1] The letter in parentheses denotes the order of an equation in
the set.

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$$H(H^2a^2\rho^2 - \psi_\eta^2)\psi_{\xi\xi} + 2H\psi_\xi\psi_\eta\psi_{\xi\eta} + H(H^2a^2\rho^2 - \psi_\xi^2)\psi_{\eta\eta}$$
$$- (\psi_\xi^2 + \psi_\eta^2)(H_\xi\psi_\xi + H_\eta\psi_\eta) = 0 \qquad\qquad (8)$$

The determinant for Equation (8) is

$$D^2 = -4H^4a^4\rho^4(1 - M^2) \qquad\qquad (9)$$

when M < 1, Equation (8) becomes an elliptical equation; for M >
1, it is a hyperbolic equation.

When H=1, Equation (8) can be simplified to obtain the equation for the stream function in orthogonal linear coordinates [94]

$$(a^2\rho^2 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (a^2\rho^2 - \psi_x^2)\psi_{yy} = 0 \qquad\qquad (10)$$

The flow equation, Equation (8), given in this paper is valid
in any conformal curvilinear coordinate system. The different con-
formal curvilinear coordinates will only differ in the expression
for the amplification factor H in Equation (8). Thus, the compli-
cated relation between the coordinates is included into the ampli-
fication factor H, allowing more complicated relations between the
coordinates to be used in practical applications. The nozzle coor-
dinates used in Section V are an example. Compared to the poten-
tial function, the advantage of using the equation for the stream
function is that the stream function is a constant at the boundar-
ies and is thus easier to treat. We will discuss this in greater
detail in the paper.

III. THE SIMPLIFIED FLOW EQUATION AND ITS SOLUTION

In two-dimensional flows, take the incompressible flow line
and the corresponding equipotential line as curvilinear coordi-
nates. In such coordinate systems, u>>v, or $|\psi_\eta|>>|\psi_\xi|$. Accord-
ing to Equation (8), after elimination of ψ_ξ and its derivatives
w.r.t.$\xi.\eta$, one obtains the simplified flow equation

$$H^2a^2\rho^2\psi_{\eta\eta} - H_\eta\psi_\eta^2 = 0 \qquad\qquad (11)$$

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Equation (11) is a nonlinear second order ordinary differential equation in η, with a parametric variable ξ. Eliminate ψ_ξ^2 from Equation (7a) to solve for H^2. Substitute this into Equation (11) to obtain the first integral of the simplified flow equation, (Equation (11), as

$$\psi_\eta = g(\xi)\rho(\xi, \eta) \tag{12}$$

in which $g(\xi)$ is related to the property of flow and the amount of flow. It is a function of ξ and is called flow-related function for simplicity. In flow around an object, it is related to the Mach number of the oncoming stream; in internal flows, it is connected to the amount of flow passing through the duct. $g(\xi)$ is determined from boundary conditions.

Using $\eta=0$ as zero streamline, i.e., when $\psi(\xi,0)$, the general solution for $\psi(\xi,\eta)$ can be obtained from Equations (4a) and (12):

$$\psi(\xi, \eta) = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} g(\xi) \int_0^\eta \left[1 - \frac{\gamma-1}{\gamma+1} \frac{g^2(\xi)}{H^2(\xi,\eta)}\right]^{\frac{1}{\gamma-1}} d\eta \tag{13}$$

For a given conformal curvilinear coordinate system, the amplification factor $H(\xi,\eta)$ is a known function of ξ,η. In Equation (13), only the flow-related function $g(\xi)$ is an unknown. It is determined by boundary conditions and will be discussed in the next section. The solution for $\psi(\xi,\eta)$ given in Equation (13) is valid for any value of the ratio of specific heats γ

Let $\gamma = 1 + \frac{1}{\nu}$. When ν is a positive whole number, substitute this expression into Equation (13) and expand the function to be integrated. Equation (13) can then be written as,

$$\left.\begin{aligned}
&\psi(\xi, \eta) = \sum_{k=0}^{\nu} A_k E_k(\xi, \eta) g^{2k+1}(\xi), \quad E_k(\xi, \eta) = \int_0^\eta H^{-2k}(\xi, \eta) d\eta \\
&A_k = (-1)^k C_k^\nu \frac{(2\nu+1)^{\nu-k}}{(2\nu)^\nu}, \quad C_k^\nu = \frac{1}{k!} \nu(\nu-1)\cdots(\nu-k+1) \\
&\nu = 1, 2, 3, \cdots; \quad k = 0, 1, 2, \cdots; \quad \nu \geq k
\end{aligned}\right\} \tag{14}$$

For a given set of conformal curvilinear coordinates, ξ, η, $E_k(\xi, \eta)$ is a known function of the coordinates. For convenience, we call it the function relating the transformations. A_k is a function of γ only. Thus, the stream function $\psi(\xi, \eta)$ becomes a polynomial in $g(\xi)$ of the $(2\nu + 1)$th degree, that has coefficients products of A_k and $E_k(\xi, \eta)$ and that contains only the odd-powered terms.

For convenience in applications, we list the following expressions for $\psi(\xi, \eta)$ for several values of γ.

$$\gamma = 2 \quad \psi(\xi, \eta) = \frac{3}{2} E_0(\xi, \eta) g(\xi) - \frac{1}{2} E_1(\xi, \eta) g^3(\xi)$$

$$\gamma = 1.5 \quad \psi(\xi, \eta) = \frac{25}{16} E_0(\xi, \eta) g(\xi) - \frac{5}{8} E_1(\xi, \eta) g^3(\xi) + \frac{1}{16} E_2(\xi, \eta) g^5(\xi)$$

$$\gamma = 1.33 \quad \psi(\xi, \eta) = \frac{343}{216} E_0(\xi, \eta) g(\xi) - \frac{49}{72} E_1(\xi, \eta) g^3(\xi)$$

$$+ \frac{7}{72} E_2(\xi, \eta) g^5(\xi) - \frac{1}{216} E_3(\xi, \eta) g^7(\xi)$$

$$(15)$$

IV. MAXIMUM FLOW AND FLOW CHARACTERISTICS OF NOZZLE

In nozzle flow, choose the incompressible potential flow line and the corresponding equipotential line as coordinates ξ, η. Considering that the flow is symmetrical w.r.t. the central flow line, we will only discuss the flow in the upper half of the nozzle. Under these conditions, the boundary conditions for the nozzle can be expressed as follows:

$$\eta = 0 时 \quad \psi(\xi, 0) = 0$$
$$\eta = \eta_b 时 \quad \psi(\xi, \eta_b) = \psi_b \bigg\} \quad (16)$$

In the above, ψ_b is the amount of flow passing through the upper half of the nozzle. Because of the limit set by the cross-section of throat in nozzle flow, ψ_b cannot be any arbitrary value, but has a maximum value which we call ψ_s.

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For the sake of simplicity, in what follows, we discuss the case where the cross-section of the throat is also symmetrical. We will see later that the results can be extended to the case of unsymmetrical throat cross-section. In the nozzle flow corresponding to symmetrical throat cross-section, the coordinate ξ for the cross-section of the throat is 0. From boundary conditions, Equation (16) and the solution for the stream function $\psi(\xi,\eta)$, Equation (13), one obtains for the amount of flow passing through the cross-section of the throat, ψ_b, the expression

$$\psi_b = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} g(0) \int_0^{\eta_b} \left[1 - \frac{\gamma-1}{\gamma+1} \frac{g^2(0)}{H^2(0,\eta)}\right]^{\frac{1}{\gamma-1}} d\eta \qquad (17)$$

In Equation (17), after the incompressible flow is specified, $H(0,\eta)$ is a definite known function. Therefore, ψ_b becomes a function of $g(0)$ only. When ψ_b is at its maximum, one has $[d\psi_b/dg(0)] = 0$. We, therefore, differentiate Equation (17) with respect to $g(0)$ to get

$$\int_0^{\eta_b} \left[1 - \frac{g^2(0)}{H^2(0,\eta)}\right]\left[1 - \frac{\gamma-1}{\gamma+1} \frac{g^2(0)}{H^2(0,\eta)}\right]^{\frac{2-\gamma}{\gamma-1}} d\eta = 0 \qquad (18)$$

In Equation (18), $g(0)$ is multi-valued; $g_s(0)$ represents the smallest positive real root and is the flow-related function corresponding to the maximum flow ψ_s passing through the cross-section of the throat. Substituting $g_s(0)$ into Equation (17), one obtains an expression for the maximum flow ψ_s as

$$\psi_s = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} g_s(0) \int_0^{\eta_b} \left[1 - \frac{\gamma-1}{\gamma+1} \frac{g_s^2(0)}{H^2(0,\eta)}\right]^{\frac{1}{\gamma-1}} d\eta \qquad (19)$$

In the nozzle flow, when ψ_b is smaller than ψ_s, one is dealing with subsonic nozzle flow, i.e., except for the local supersonic flow that might appear near the throat wall, the flow on both sides of the throat is subsonic. When ψ_b equals ψ_s, one has supersonic nozzle flow. In other words, under this condition, the flow

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in the throat region turns from subsonic into supersonic. The attainment of maximum amount of flow ψ_s in the nozzle is a necessary condition for guaranteeing continuity of flow in the throat region and avoiding singular points. The determination of maximum flow ψ_s is thus a very important problem.

Let the boundary conditions be such that $\psi_b \leq \psi_s$. After ψ_b is given, one can use the boundary conditions to obtain from Equation (13) the integral equation for determining the flow-related function $g(\xi)$

$$\psi_b = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} g(\xi) \int_0^{\eta_b} \left[1 - \frac{\gamma-1}{\gamma+1} \frac{g^2(\xi)}{H^2(\xi,\eta)}\right]^{\frac{1}{\gamma-1}} d\eta \qquad (20)$$

This is an integral equation for $g(\xi)$ with ξ as a parametric variable. For a given ξ, $g(\xi)$ in Equation (20) is multi-valued. To ensure continuity of solution and that the density remains positive, the value for $g(\xi)$ should be chosen in the following manner: For subsonic flow, $\psi_b < \psi_s$, take $g(\xi)$ to be the smallest positive real root for the entire range of ξ. For supersonic nozzle flow, $\psi_b = \psi_s$, the choice of $g(\xi)$ is determined by the value of ξ. When $\xi < 0$, take $g(\xi)$ to be the smallest positive real root; this corresponds to the flow in the contraction section. When $\xi = 0$, take $g(\xi)$ to be the smallest positive real multiple root; this corresponds to the flow in the cross-section of the throat. $\xi > 0$, take $g(\xi)$ to be the second real root; this corresponds to the flow in the divergent section.

When $\gamma = 1 + \frac{1}{\nu}$ and ν is a positive whole number, as $\psi(\xi,\eta)$ can be expressed as a polynomial, the corresponding maximum flow ψ_s and the flow-related function $g(\xi)$ can be calculated using polynomials. Thus, the amount of flow passing through the cross-section of the throat ($\xi = 0$) can be expressed as

$$\psi_s = \sum_{k=0}^{\nu} A_k E_k(0, \eta_b) g^{2k+1}(0) \qquad (21)$$

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Differentiate Equation (21) w.r.t. g(0) and set the result to zero. One obtains the algebraic equation satisfied by $g(0)$ at maximum flow:

$$\sum_{k=0}^{\bullet} (2k + 1) A_k E_k(0, \eta_b) g^{2k}(0) = 0 \qquad (22)$$

Let $g_s(0)$ represent the smallest positive real root at g(0) in Equation (22) and substitute it into Equation (21) to obtain ψ_s

$$\psi_s = \sum_{k=0}^{\bullet} A_k E_k(0, \eta_b) g_s^{2k+1}(0) \qquad (23)$$

The relation used to determine g(ξ) from boundary conditions is given by

$$\sum_{k=0}^{\bullet} A_k E_k(\xi, \eta_b) g^{2k+1}(\xi) = \psi_b \qquad (24)$$

In Equation (24), g(ξ) is multi-valued. The choice of g(ξ) under various conditions is the same as in Equation (20).

The above polynomial form for the stream function $\psi(\xi, \eta)$ is applicable for the cases where γ is 2, 1.5, 1.33, 1.25, etc. Although the choice of γ is thus limited, the method of solution is simple and easy to apply. For example, when γ is 2, one obtains from Equations (14), (16), (22), (23), (24), the following analytical expressions for $g(\xi)$, $g_s(0)$ and ψ_s:

$$\left.\begin{aligned}
g(\xi) &= [2g_s(0)/\omega(\xi)] \cos\frac{1}{3} \{\pi \pm \cos^{-1}[k\omega(\xi)]\} \\
\omega(\xi) &= [E_1(\xi, \eta_b)/E_1(0, \eta_b)]^{\frac{1}{2}} \\
g_s(0) &= [\eta_b/E_1(0, \eta_b)]^{\frac{1}{2}} \\
\psi_s &= \eta_b g_s(0), \quad k = \psi_b/\psi_s
\end{aligned}\right\} \qquad (25)$$

In conformal curvilinear coordinates, because the Lamé coefficients are equal, the relation for the characteristic line is the same as that in orthogonal linear coordinates. Thus we have

$$\frac{d\xi}{d\eta} = \text{ctg}\,(\delta \pm \mu) \,, \delta = -\text{tg}^{-1}(\psi_\xi/\psi_\eta), \quad \mu = \sin^{-1}\frac{1}{M} \qquad (26)$$

Usually, the characteristic lines obtained by taking the positive sign in Equation (26) are called the first family of characteristic lines, and those obtained by taking the negative sign are said to belong to the second family.

For convenience, the second family characteristic line that passes through the point $(\xi = 0, \eta = \eta_s)$ on the cross-section of the throat is called the influence line. From the characteristics of supersonic flow, the flow upstream of the influence line will not be affected by the wall surface of the divergent section.

The second family characteristic line that passes through the sound velocity point on the center line $(\xi = \xi_s, \eta = 0)$ is called the limiting characteristic line. The flow downstream the limiting characteristic line is generally considered not to affect the shape of the sound velocity line. In nozzle flows, the wall point of the limiting characteristic line is upstream the cross-section of the throat, and the velocity line is affected by the maximum flow ψ_s, which is dependent on the shape of the throat wall at the cross-section. Hence, when adjusting the wall shape of the throat downstream the limiting characteristic line, one should keep ψ_s unchanged so as to ensure that the shape of the velocity line stays the same.

Usually, the first family characteristic line that passes through the sound velocity point on the center line is called the branchline. When using the hodograph method to discuss nozzle flow, the branchline is a singular line of the hodograph. It satisfies the relation $[\partial(q, 0)/\partial(\phi, \psi)] = 0^{[27]}$.

To find the time $\tau(s, \psi)$ spent by a gas mass point on any stream line, let $s = s_0$ be the zero reference time, i.e., $\tau(s_0, \psi) = 0$. Then the time spent along any stream line $\psi = C$ can be written as

$$\tau(s, \psi) = \int_{s_0}^{s} \frac{ds}{q(s, \psi)}, \quad s(\xi, \eta) = \int_{\xi_0}^{\xi} H(1 + \psi_\xi^2/\psi_\eta^2)^{\frac{1}{2}} d\xi \qquad (27)$$

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in which ξ_0 is the value of ξ on each streamline corresponding to s_0. $\tau(s,\psi)$ is dependent on the geometric shape of the nozzle. In the design of fast acceleration short nozzles, the wall contours of the nozzle selected has to satisfy the requirement of minimum $\tau(s,\psi)$.

V. NOZZLE COORDINATES

According to the theory of complex variables and that of in-compressible flow, choose symmetrical hyperbolic wing shape in the incompressible flow velocity plane and obtain a symmetrical nozzle in the physical z plane. The transformation relationship equation $f(\zeta)$ between z plane and complex potential ζ plane is given by

$$z = \frac{n}{\eta_b}\left\{\zeta - (1-\sigma)\coth\lambda\ln\left[\frac{\mathrm{ch}\frac{1}{2}(\lambda+\zeta)}{\mathrm{ch}\frac{1}{2}(\lambda-\zeta)}\right]\right\} \tag{28}$$

in which $\lambda, \sigma,\ n_b$ are shape factors of the nozzle, and n is the con-traction ratio of the nozzle. Details will be given in another paper. Separate the real and imaginary parts of Equation (28), and obtain the following relation between the x,y coordinates and the ξ,η coordinates:

$$x(\xi,\eta) = \frac{n}{\eta_b}\left\{\xi - \frac{1}{2}(1-\sigma)\coth\lambda\ln\left[\frac{\mathrm{ch}(\lambda+\xi)+\cos\eta}{\mathrm{ch}(\lambda-\xi)+\cos\eta}\right]\right\}$$
$$y(\xi,\eta) = \frac{n}{\eta_b}\left[\eta - (1-\sigma)\coth\lambda\,\mathrm{tg}^{-1}\left(\frac{\mathrm{sh}\lambda\sin\eta}{\mathrm{ch}\xi+\mathrm{ch}\lambda\cos\eta}\right)\right] \tag{29}$$

From Equations (6d), (28) or (29), one obtains the amplifi-cation factor $H(\xi,\eta)$ of z plane w.r.t. ζ plane

$$H(\xi,\eta) = \frac{n}{\eta_b}\left[\frac{(\cos\eta+\sigma\,\mathrm{ch}\lambda\,\mathrm{ch}\xi)^2 - (\sigma^2\mathrm{ch}^2\lambda-1)\mathrm{sh}^2\xi}{(\cos\eta+\mathrm{ch}\lambda\,\mathrm{ch}\xi)^2 - \mathrm{sh}^2\lambda\,\mathrm{sh}^2\xi}\right]^{\frac{1}{2}} \tag{30}$$

When $\eta = \eta_b$, Equation (29) gives the contour line of the wall of a symmetrical nozzle, with ξ as a parametric variable. From analytical geometry, the following relation exists among the nozzle wall shape parameters r (radius of curvature of throat wall),

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n (contraction ratio), θ_1 (largest angle of inclination of wall) and $\xi_\perp$ (point of inflection at the wall):

$$r = [(1 + y')^{3/2}/y'']_{r=0}, \quad n = y(\infty, \eta_b)$$
$$\theta_1 = \text{tg}^{-1} y'(\xi_1, \eta_b), \quad y''(\xi_1, \eta_b) = 0 \tag{31}$$

In the above equations, y' and y" represent the first and second derivative respectively of y w.r.t. x. From Equations (30), (31), and $n = n_b$, we obtain the following relations between the wall shape parameters and λ, σ, n_b:

$$r = \frac{n}{\eta_b} \frac{(\cos\eta_b + \sigma\,\text{ch}\,\lambda)^2}{(1-\sigma)\,\text{ch}\,\lambda\sin\eta_b},$$

$$n = \left[1 - \frac{1-\sigma}{\eta_b}\coth\lambda\,\text{tg}^{-1}\left(\frac{\text{sh}\,\lambda\sin\eta_b}{1+\text{ch}\,\lambda\cos\eta_b}\right)\right]^{-1}$$

$$\theta_1 = \text{tg}^{-1}\left[\frac{(1-\sigma)\,\text{ch}\,\lambda\sin\eta_b\,\text{sh}\,\xi_1}{\text{ch}^2\,\xi_1 + (1+\sigma)\,\text{ch}\,\lambda\cos\eta_b\,\text{ch}\,\xi_1 + \sigma\,\text{ch}^2\,\lambda - \sin^2\eta_b}\right]$$

$$\text{ch}\,\xi_1 = \left[\frac{3}{4}(\sigma\,\text{ch}^2\,\lambda + \cos^2\eta_b + 1)\right]^{\frac{1}{2}} \times \cos\frac{1}{3}\left\{\cos^{-1}\frac{4(1+\sigma)\,\text{ch}\,\lambda\cos\eta_b}{\left[\frac{3}{4}(\sigma\,\text{ch}^2\,\lambda + \cos^2\eta_b + 1)\right]^{\frac{3}{2}}}\right\} \tag{32}$$

With given nozzle shape parameters r, n, θ_1, one can obtain from the above equation the corresponding value for λ, σ, n_b and hence, the desired nozzle shape line.

VI. $E_k(\xi, \eta)$, FUNCTION RELATING NOZZLE TRANSFORMATION

For the above nozzle which had adjustable radius of curvature r, contraction ratio n and largest angle of inclination of wall θ_1, when $\gamma = 1 + \frac{1}{\nu}$, where ν is a positive whole number, the corresponding function relating to the transformation, $E_k(\xi, \eta)$ can be obtained in the form of an analytical expression by integration. Substitute Equation (30) into Equation (14b), express the integrand in terms of partial fractions, integrate separately and sum together to obtain $E_k(\xi, \eta)$:

$$E_k(\xi, \eta) \sim \left(\frac{\eta_\Lambda}{n}\right)^{2k} \sum_{l=0}^{k} C_l^k \left(\frac{1-\sigma}{\sigma}\right)^l G_l(\xi, \eta)$$

$$G_0(\xi, \eta) = \eta$$

$$G_l(\xi, \eta) = \sum_{m=1}^{l} [B_{lm}(\xi, \alpha)N_m(\xi, \eta, \alpha) + B_{lm}(\xi, -\alpha)N_m(\xi, \eta, -\alpha)]$$

$$l = 1, 2, 3, \cdots; l \leqslant k$$

$$C_l^k = \frac{1}{l!} k(k-1)\cdots(k-l+1)$$

$$\mathrm{ch}\,\alpha = \sigma\,\mathrm{ch}\,\lambda$$

(33)

In Equation (33), $N_m(\xi, \eta, \alpha)$ and $B_{lm}(\xi, \alpha)$ are respectively:

$$N_1(\xi, \eta, \alpha) = \frac{2}{\mathrm{th}\,\xi + \mathrm{th}\,\alpha} \mathrm{tg}^{-1}\left(\mathrm{th}\,\frac{\xi+\alpha}{2} \cdot \mathrm{tg}\,\frac{\eta}{2}\right)$$

$$N_m(\xi, \eta, \alpha) = \frac{1}{(m+1)(\mathrm{th}\,\xi + \mathrm{th}\,\alpha)^2}\Big[(2m-3)(1+\mathrm{th}\,\alpha)\,\mathrm{th}\,\xi N_{m-1}(\xi, \eta, \alpha)$$

$$- (m-2)N_{m-2}(\xi, \eta, \alpha) - \frac{\mathrm{sech}\,\alpha\,\mathrm{sech}\,\xi\,\sin\eta}{(1+\mathrm{th}\,\alpha\,\mathrm{th}\,\xi + \mathrm{sech}\,\alpha\,\mathrm{sech}\,\xi\cos\eta)^{m-1}}\Big]$$

$$B_{11}(\xi, \alpha) = 1 + \coth\alpha\coth\xi\left(\mathrm{th}^2\xi - \frac{1-\sigma}{2\sigma}\mathrm{sech}^2\xi\right)$$

$$B_{lm}(\xi, \alpha) = \sum_{i=m}^{l} (-1)^{l-i}\frac{1}{2^{l-m}} C_i^l C_{i-m}^{i-m-1} \frac{B_{11}^i(\xi, \alpha)B_{11}^{l-i}(\xi, -\alpha)}{(\mathrm{th}\,\alpha\,\mathrm{th}\,\xi)^{l-m}}$$

$$l, m = 1, 2, 3, \cdots; m \leqslant l$$

(34)

For convenience, the expressions for $B_{lm}(\xi, \alpha)$ for $l, m \leqslant 3$ are found from Equation (34d) and listed below. These correspond to the cases where $\gamma = 1.5$ and 1.33:

$$B_{21}(\xi, \alpha) = -\coth\alpha\coth\xi B_{11}(\xi, \alpha)B_{11}(\xi, -\alpha)$$

$$B_{22}(\xi, \alpha) = B_{11}^2(\xi, \alpha)$$

$$B_{31}(\xi, \alpha) = \frac{3}{4}\coth\alpha\coth\xi B_{21}(\xi, \alpha)[B_{11}(\xi, \alpha) - B_{11}(\xi, -\alpha)]$$

$$B_{32}(\xi, \alpha) = \frac{3}{2} B_{21}(\xi, \alpha) B_{11}(\xi, \alpha)$$

$$B_{33}(\xi, \alpha) = B_{11}^3(\xi, \alpha)$$

(35)

At the cross-section of the throat, $\xi = 0$. Use this relation in Equations (33) and 34b); integrate to obtain the less complicated expression for $E_k(0,\eta)$:

$$
\left.
\begin{aligned}
&E_k(0, \eta) = \left(\frac{\eta_b}{\eta}\right)^{2k} \sum_{m=0}^{2k} C_m^{2k} \left(\frac{1-\sigma}{\sigma}\right)^m N_m(0, \eta), \quad k = 0, 1, 2, \cdots \\[2mm]
&N_0(0, \eta) = \eta, \quad N_1(0, \eta) = 2\coth\alpha\, \mathrm{tg}^{-1}\left(\mathrm{th}\frac{\alpha}{2} \cdot \mathrm{tg}\frac{\eta}{2}\right) \\[2mm]
&N_m(0, \eta) = \frac{\mathrm{ch}^{m-2}\alpha}{m-1} \coth^2\alpha \left[(2m - 3)N_{m-1}(0, \eta) \right. \\[2mm]
&\qquad\qquad\left. - (m - 2)N_{m-2}(0, \eta) - \frac{\mathrm{sech}\,\alpha\sin\eta}{(1 + \mathrm{sech}\,\alpha\cos\eta)^{m-1}}\right] \\[2mm]
&m = 2, 3, \cdots; \; m \leqslant k
\end{aligned}
\right\}
\tag{36}
$$

Usually, $\sigma\mathrm{ch}\lambda > 1$, and $\mathrm{ch}\alpha$ is greater than 1, where α is a real number. When the radius of curvature of the throat wall is very small, for instance, when $\eta_b = \pi/2$, $r = 0.3$, $\sigma\mathrm{ch}\lambda < 1$. In this case, α is an imaginary number. The corresponding $N_m(\xi,\eta,\alpha)$ and $B_{1m}(\xi,\alpha)$ are complex numbers. Considering that $H(\xi,\eta)$ is always a positive real number, $E_k(\xi,\eta)$ must also be a positive real number. When α is imaginary, let $\alpha = i\beta$, in which β is a real number. Then one has $\mathrm{ch}\alpha = \cos\beta$, $\mathrm{sh}\alpha = i\sin\beta$, and the identity

$$
\begin{aligned}
\mathrm{tg}^{-1}\left(\mathrm{th}\frac{\xi + i\beta}{2} \cdot \mathrm{tg}\frac{\eta}{2}\right) &= \frac{1}{2}\,\mathrm{tg}^{-1}\left(\frac{\sin\eta\,\mathrm{th}\,\xi}{\cos\eta + \cos\beta\,\mathrm{sech}\,\xi}\right) \\[2mm]
&+ \frac{i}{2}\,\mathrm{th}^{-1}\left(\frac{\sin\eta\,\mathrm{tg}\,\beta}{\cos\eta + \sec\beta\,\mathrm{ch}\,\xi}\right)
\end{aligned}
\tag{37}
$$

Thus, using the above identify and equations to separate $N_1(\xi,\eta,\alpha)$ and $B_{11}(\xi,\alpha)$ into real and imaginary parts and writing $N_m(\xi,\eta,\alpha)$ and $B_{1m}(\xi,\alpha)$ as functions of $N_1(\xi,\eta,\alpha)$ and $B_{11}(\xi,\alpha)$, after the results for $N_m(\xi,\eta,\alpha)$ and $B_{1m}(\xi,\alpha)$ are substituted into Equation (33), one obtains the corresponding $G_1(\xi,\eta)$ and $E_k(\xi,\eta)$. Furthermore, when $\sigma\mathrm{ch}\lambda < 1$, one can use the integral Equations (13) and (17)-(20) to obtain the results directly.

VII. AN EXAMPLE OF NOZZLE FLOW FIELD COMPUTATION

In this section, we show computations based on the general solution for nozzle flow given above for the simplest case, i.e., when γ equals 2. From Equations (15) and (25), one obtains for $\gamma = 2$ the simple analytical expression for $\psi(\xi,\eta)$:

$$
\begin{aligned}
\psi(\xi,\eta) &= [\,3\eta g(\xi) - E_1(\xi,\eta)g^3(\xi)\,]/2 \\
g(\xi) &= 2\eta_b^{1/2}E_1^{-1/2}(\xi,\eta_b)\cos\frac{1}{3}\{\pi \pm \cos^{-1}[kE_1^{-1/2}(0,\eta_b)E_1^{1/2}(\xi,\eta_b)]\} \\
E_1(\xi,\eta) &= \left(\frac{\eta_b}{n}\right)^2\eta + \frac{1-\sigma}{\sigma}\left(\frac{\eta_b}{n}\right)^2[\,B_{11}(\xi,\alpha)N_1(\xi,\eta,\alpha) \\
&\quad + B_{11}(\xi,-\alpha)N_1(\xi,\eta,-\alpha)] \\
B_{11}(\xi,\alpha) &= 1 + \coth\alpha\coth\xi\left(\mathrm{th}^2\,\xi - \frac{1-\sigma}{2\sigma}\,\mathrm{sech}^2\,\xi\right) \\
N_1(\xi,\eta,\alpha) &= \frac{2}{\mathrm{th}\,\xi + \mathrm{th}\,\alpha}\,\mathrm{tg}^{-1}\left(\mathrm{th}\frac{\xi+\alpha}{2}\cdot\mathrm{tg}\frac{\eta}{2}\right) \\
\psi_i &= \eta_b^{1/2}E_1^{-1/2}(0,\eta_b),\quad k = \psi_b/\psi_i,\quad \mathrm{ch}\,\alpha = \sigma\,\mathrm{ch}\,\lambda
\end{aligned}
\right\}\quad (38)
$$

Under these conditions, $\rho(\xi,\eta)$ in Equation (7a) can be expressed as

$$
\rho(\xi,\eta) = \frac{1}{2}\cos\left\{\frac{1}{3}\cos^{-1}\left[1 - \frac{2}{H^2(\xi,\eta)}(\psi_\xi^2 + \psi_\eta^2)\right]\right\}\quad (39)
$$

In Equation (39), $\cos^{-1}\left[1 - \dfrac{2}{H^2(\xi,\eta)}(\psi_\xi^2 + \psi_\eta^2)\right]$ has values between 0 and π in the subsonic range, and between π and 2π in the supersonic range.

The flow for the case in which $\gamma = 2$ is equivalent to the motion of gas which, in the water-table experiment, was simulated with the free motion on the water surface [93]. For the cases where γ equals 1.5, 1.33, 1.25, 1.20,..., the stream function $\psi(\xi,\eta)$ has analytical expressions similar to Equation (38); for any other values of γ, one can use the integral expressions Equations (13)

and (29). As we are limited in space here, the results for computation for other values of γ will be given elsewhere.

When $ch\alpha = \sigma ch\lambda < 1$, the corresponding $B_{11}(\xi,\alpha)$ and $N_1(\xi,\eta,\alpha)$ in Equation (38d,e) are complex numbers. In this case, let $\alpha = i\beta$, use Equation (37) to separate $B_{11}(\xi,\alpha)$ and $N_1(\xi,\eta,\alpha)$ into real and imaginary parts, and substitute into Equation (38c). One then obtains the corresponding expression for $E_1(\xi,\eta)$:

$$
E_1(\xi,\eta) = \left(\frac{\eta_b}{n}\right)^2 \left(\frac{1-\sigma}{\sigma}\right) (tg^2\beta + th^2\xi)^{-1}\left[\frac{1-\sigma}{\sigma}(1-th^2\xi)\right.
$$
$$
\times \coth\xi \, tg^{-1}\left(\frac{\sin\eta\,th\,\xi}{\cos\eta + \cos\beta\,sech\,\xi}\right) + \left(2\sec^2\beta - \frac{1+\sigma}{\sigma}sech^2\xi\right)ctg\beta
$$
$$
\left.\times th^{-1}\left(\frac{\sin\eta\,tg\beta}{\cos\eta + \sec\beta\,ch\,\xi}\right)\right] + \left(\frac{\eta_b}{n}\right)^2\eta \tag{40}
$$

One can use Equations (38), (39), (7b), (4b) and Equations (26) and (27) to obtain the density $\rho(\xi,\eta)$ in the flow field, velocities $u(\xi,\eta)$, $v(\xi,\eta)$ and Mach number $M(\xi,\eta)$ distribution, along with the influence line, limiting characteristics line, branchline and equitemporal lines.

Note that the stream function given in Equation (38) contains three shape factors λ,σ,η_b and one flow factor k. For given nozzle wall shape yarameters, r, n, and θ_1, one can determine the corresponding shape factors λ,σ,η_b from Equation (32). Flow factor k varies in the range between 0 and 1. k less than 1 gives subsonic nozzle flow, while k equal to 1 gives supersonic nozzle flow.

Computation for a typical nozzle is given below. The shape parameters of the nozzle are taken to be, respectively, r=2, n=2, θ_1=23.3°; the corresponding shape factors are, respectively, λ=2.23, σ=0.434, $\eta_b = \frac{\pi}{2}$. The calculations are done for k equal to 0.50, 0.90, 0.98 and 1.00, respectively. These correspond to lower subsonic, medium subsonic, higher subsonic and supersonic nozzle flow, respectively. The results of the calculations are shown in Figures 1-6.

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Figure 1 gives the distribution of lines of equal Mach number for subsonic nozzle flow (k=0.50). The largest Mach number in the figure is 0.335. Hence, we can consider the flow to be incompressible.

Figure 2 gives the distribution of lines of equal Mach number for medium subsonic nozzle flow (k=0.90). It can be seen from the figure that the Mach number M>0.7 near the throat wall. Hence, one must take into consideration the effect of compressibility in this region.

Figure 3 gives the distribution of lines of equal Mach number for higher subsonic nozzle flow (k=0.98). It can be seen from the figure that local supersonic flow appears near the throat wall. This type of flow also belongs to the realm of transonic flow problems.

Figure 4 gives the distribution of lines of equal Mach number for supersonic nozzle flow (k=1). It also shows the influence line, limiting characteristic line and branchline. AE represents the sound velocity line. Note that the sound velocity point A at the wall is located upstream of the cross-section of the throat and the sound velocity point E on the center line is located downstream of the cross-section of the throat. BE is the limiting characteristic line. It is the second family characteristic line that passes through point E. Wall point B is located upstream of point C on the cross-section of the throat and downstream sound velocity point A. It is usually regarded that the flow field downstream of the limiting characteristic line and the shape of the wall will not affect the shape of the sound velocity line.

In nozzle flows, the entire flow is affected by the maximum flow ψ_s, which in turn, is dependent on the shape of the throat wall. Hence, to ensure that the shape of the sound velocity line will not vary, one must make sure that changing the wall shape
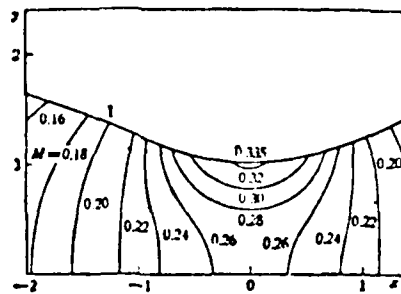
Figure 1.   Distribution of lines of equal Mach number
for lower subsonic nozzle flow

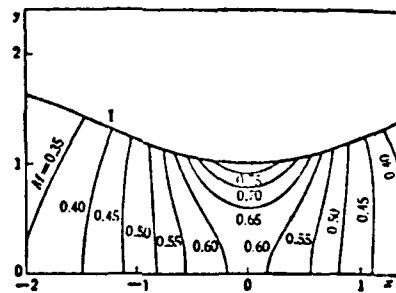$n=2.0$, $r=2.0$, $\theta_1=23.3°$, $\gamma=2.0$, $k=0.50$

Figure 2.   Distribution of lines of equal Mach number
for medium subsonic nozzle flow

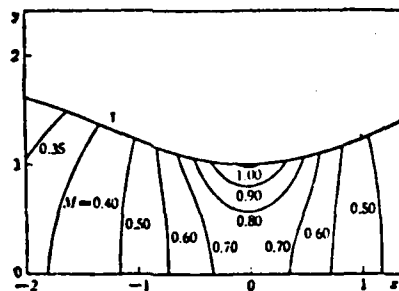$n=2.0$, $r=2.0$, $\theta_1=23.3°$, $\gamma=2.0$, $k=0.90$

Figure 3.   Distribution of lines of equal Mach number
for higher subsonic nozzle flow

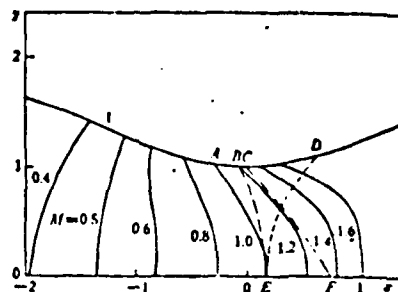$n=2.0$, $r=2.0$, $\theta_1=23.3°$, $\gamma=2.0$, $k=0.98$

Figure 4.   Characteristic lines for supersonic nozzle flow

$n=2.0$, $r=2.0$, $\theta_1=23.3°$, $\gamma=2.0$, $k=1.0$

———lines of equal Mach number sound velocity line AE
–··–·–limiting characteristic line
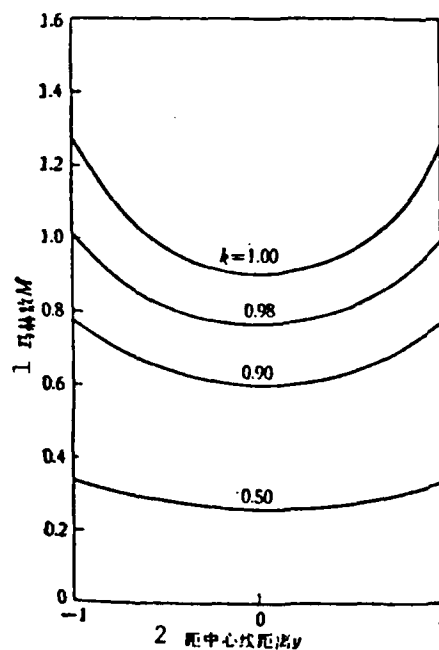– ·· –··influence line CF
–––– branchline DE

Figure 5.   Distribution of Mach number at cross-section of throat

$n=2.0$, $r=2.0$, $\theta_1=23.3°$, $\gamma=2.0$

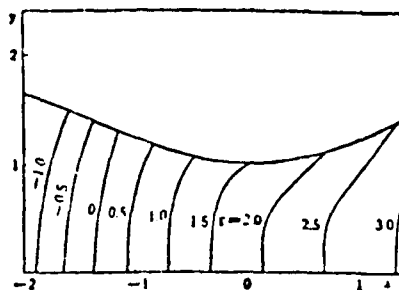1--Mach number; 2--distance from center line, y

Figure 6. Equitemporal lines of supersonic nozzle flow

$n=2.0$, $r=2.0$, $\theta_1=23.3°$, $\gamma=2.0$, $k=1.0$

downstream the limiting characteristic line will not change the maximum flow ψ_s. CF is the influence line, i.e., the second family characteristic line that passes through wall point C at the throat. The flow downstream CF will not affect the flow in the convergent section. In other words, the flow upstream CF is not affected by the wall of the divergent section. Therefore, one can choose a line downstream of the sound velocity line and upstream of the influence line as the initial condition for the design of the divergent section. According to the requirements of the initial value problem, this line cannot be a characteristic line. ED is the branchline, the first family characteristic line that passes through the sound velocity point on the center line. When using the hodograph method to analyze nozzle flow, the branchline is a singular line on the hodograph plane [27].

Figure 5 gives the distribution of Mach number in the cross-section of the throat for different amounts of flow. From the figure, one can see that the greater the flow factor k, the less uniform the distribution of Mach number. Hence, the assumption of a linear sound velocity line in design of supersonic nozzles [83] is not reasonable. One must take into consideration the effect of a curvilinear sound velocity line, especially for the case of smaller radius of curvature of throat wall.

Figure 6 gives the equitemporal lines of supersonic nozzle flow. The zero reference time is taken at $\xi = \xi_1$. From the figure, one can see that the density of equitemporal lines is higher in the subsonic region, i.e., the gas spends more time in the subsonic region. Hence, to hasten the flow of gas in the nozzle so as to increase the freezing efficiency of high-temperature gases under non-equilibrium conditions, one must give heed in the design of the contour lines of the wall in the convergent section.

VIII. DISCUSSIONS

1) The solution for subsonic, transonic and supersonic nozzle flow has been given in a generalized form. It is applicable to lower subsonic, medium subsonic, higher subsonic and supersonic nozzle flows.

2) We have given a nozzle whose wall shape can be chosen as desired and its solution. The wall shape of the nozzle is determined by three parameters: radius of curvature r of the throat wall, contraction ratio n and largest angle of inclination θ_1 of wall. r can be taken to be any value from 0.05 to infinity and n can have any value from 1 to infinity. In choosing θ_1, the upper limit increases with increasing contraction ratio n, and increases with decreasing radius of curvature r of throat wall. For example, when r=1, and n is 2,5,10,20, the upper limit for θ_1 is respectively 23°, 45°, 57°, 60.5°.

3) The solution for nozzle flow given is valid for any value of the ratio of specific heats γ. Under general conditions, the solution for the stream function $\psi(\xi,\eta)$ is in integral form. When $\gamma = 1 + (1/\nu)$, where ν is a positive whole number, $\psi(\xi,\eta)$ takes the form of an odd-powered polynomial.

4) The expression for the maximum flow ψ_s has been given. ψ_s is a function of the shape of the wall. When the radius of curvature

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r of the throat wall is large, $\psi_s \overset{\sim}{\sim} 1$. With decreasing r, ψ_s decreases.

5) The results of calculation show that when r is small, the sound velocity line is far from being linear. Under these conditions, it is not reasonable to use a linear line for the sound velocity line to be used as initial condition in the design of the divergent section [82,83]. A similar situation prevails in axially symmetric nozzle flows [75,76].

6) The second family characteristic line that passes through the wall point ($\xi = 0$, $\eta = \eta_b$) in the cross-section of the wall is called the influence line. From the properties of supersonic flows, one knows that the flow upstream of the influence line is not affected by the shape of the wall in the divergent section. Hence, one can take a line downstream of the sound velocity line and upstream of the influence line, specify the magnitude and direction of the velocity on this line, and use it as initial condition in the design of the divergent section. By the requirements of the initial value problem, this line cannot be one of the characteristic lines.

7) The solution given for the simplified equation for the stream function has a high degree of accuracy in the transonic region. It has also given satisfactory results for the higher subsonic region and lower supersonic region. For higher-order solutions, one can take this solution as a basis and use the general form, Equation (8), for the stream function in conformal curvilinear coordinates to do the analysis.

8) The solution given for the subsonic nozzle flow corresponds to that for symmetrical throat cross-section. One can further discuss higher order approximations, using this solution as a basis.

9) For nozzles with hyperbolic or circular-arc-shaped throat wall, the solution given for the transonic flow has a simple form

and is convenient to use in the analysis of the properties of flow in the throat region.

10) The characteristics of transonic flow in a hyperbola-shaped nozzle obtained by our method are in good agreement with those obtained by Cherry [52] and Serra [81] for the case where $\gamma = 1.4$. The results will be given elsewhere.

11) From Equation (20), it is obvious that when q is close to 1, γ has little effect on $g(\xi)$. Roughly speaking, in this case, $g(\xi)$ is dependent on the geometric shape of the nozzle only. Hence, for a given nozsle, the shape of the sound velocity line is basically unaffected by the ratio of specific heats γ.

12) To study the effect of nozzle wall shape on the flow field, particularly the relation between the flow characteristics of the transonic region at the throat and the geometric shape of the nozzle, one can use the nozzle given in this paper, do calculations using various shape parameters r,n,θ_1, and make a table for reference in the engineering design of nozzles.

13) When it is not feasible to express in simple analytical forms the incompressible potential flow of the nozzle, one can use numerical methods to calculate the streamlines and equipotential lines of the incompressible flow. On this basis, one can use the present method to compute the corresponding subsonic, transonic and supersonic nozzle flow.

14) This method can be extended to flow around objects. Here, the manner in which the flow-related function is determined from the boundary conditions differs from that in the case of internal flows. We will discuss this in another paper.

15) In analyzing axially symmetric flows, one can use the streamlines and equipotential lines of the corresponding incompressible potential flow as coordinates to solve for the axially symmetric flow of compressible gases.

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IX. CONCLUSIONS

This article gives the general form of the equation for the
two-dimensional constant steady irrotational isentropic flow of
an ideal gas in conformal curvilinear coordinates. Simplified
equation for the stream function and its general solution is given
using the streamlines and equipotential lines of the corresponding
incompressible potential flow as coordinates.

The results are applied to nozzle flow, giving the solution
for nozzle flow from subsonic through transonic to the supersonic
region, for any desired choice of the wall shape, i.e., radius of
curvature of throat wall, contraction ratio and largest angle of
inclination of wall. This solution is valid for different ratios
of specific heats. As an example, we have calculated the flow char-
acteristics of a typical nozzle.

In this paper, the relation between the maximum flow and the
shape of the wall is given. The maximum flow decreases with
decreasing radius of curvature of throat wall. For very large
radius of curvature of throat wall, e.g., r>5, the maximum flow
ψ_s can be approximated by 1.

This paper gives the flow characteristics of a nozzle for
different amounts of flow. The case with flow factor k less than
1 corresponds to subsonic nozzle flow, i.e., except for local super-
sonic region that may appear near the throat wall, the flow on both
sides of the throat is subsonic. When k equals 1, the flow is
supersonic, i.e., near the throat, the flow changes from subsonic
to supersonic.

In supersonic nozzle flows, the shape of the sound velocity
line is greatly affected by the contour of the wall, especially
the radius of curvature of throat wall. For a nozzle of given wall
contours, the shape of the sound velocity line is not much affected
by the ratio of specific heats, γ. The results of calculations

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for hyperbola-shaped nozzles show that when γ varies from 1.2 to 1.667, the sound velocity point on the center line moves downstream a little bit, the sound velocity point on the wall moves upstream a little bit, but the changes are very small. For details, see another paper to be given later.

The second family characteristic line that passes through the wall point of the cross-section of throat is called the influence line. Considering that the flow upstream of the influence line is not very much affected by the wall contours of the divergent sections, one can choose a line between the sound velocity line and the influence line, use the method given in this paper to give the magnitude and direction of velocity on this line and use this as initial condition for the design of the divergent section. Downstream the influence line, the contour line of the divergent section can be designed by means of the method of characteristic lines. By the requirements of the initial value problem, this initial line cannot be a characteristic line.

This method can be extended to flow around an obstacle over weir flow in the nozzle and flow through a grating. Particularly, one can obtain better results in the transonic region. Furthermore, the method can be extended to cases where there are chemical reactions under equilibrium or non-equilibrium conditions, or where there exists axial symmetry.

REFERENCES

The references used in this paper can be divided into two parts. The first contains 88 references related to nozzle research. Of these, 80 are related to transonic nozzle flow and eight to incompressible nozzle flow. In order to give the reader an idea of the development of nozzle research, we have arranged these 88 papers chronologically. The second contains references used in this paper that are related to other studies. There are nine such references, also arranged chronologically.

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RESEARCH ON THE HISTORY OF MECHANICS

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Through production struggle, class struggle and scientific experimentation, the working people of our country in ancient times made great contributions to mankind with their achievements in science. Our people are good at utilizing accumulated experience and skillfully making various tools and machines based on mechanical principles. They have invented many things, one of which is the rocket.

I.   THE ANCIENT "ROCKET"

"Rocket" is a term that appeared early in our country. In the concise edition if Wei Lüeh, it is recorded that when Chu Ke Liang was attacking He Chao, he "raised the tall ladders and had his men climb onto the wall. Chao used rockets to shoot at the ladder. The ladder caught fire and all the men on it were burned to death". Tu Huey Tu Chuan, vol. 92 of Sung Shu, records the battle between Tu Huey Tu and Jui Hsün on water, in which "Huey Tu personally fought on the battleship, had rockets shot onto Hsün's battleships which all caught fire. Hsün lost the battle immediately. He was shot by an arrow and died in the water". In Wang Szu Cheng Chuan, vol. 62 of Pei Shih, there is also an account of how "Szu Cheng used rockets to shoot "the attacking enemies and burned their

attacking instruments". These accounts are evidence that there
existed something called "rockets" as early as in the era of the
Three Kingdoms, and it continued to be used during the South and
North Dynasties. These "rockets" were shot out. Apparently,
those were just arrows with flammable substances added to burn
the objects and did not the kind of rocket that is propelled by
explosives.

This type of "rocket" was further developed in the Tang Dynas-
ty. In Vol. 4 of "Chen Chi Chih Ti Tai Pai Yin Ching", Li Ch'uan
recorded a kind of rocket in greater detail. "Rockets: Attach a
ladle of oil to the arrowhead and shoot the arrow at the fortress.
When the ladle breaks and the oil leaks out, follow up with a
rocket so that a fire is started. Then follow up with more oil-
ladles and the fortress is all burned down". This account may
have been taken from the Ch'ing Dynasty edition of Wei Kung Ping Fa
edited by Li Ching during the early period of the Tang Dynasty.
The wording is almost identical to that used to describe "rockets"
in Tai Pai Lin Ching [1]. This kind of rocket is nothing but an
ignited arrow. A fire starter is shot at the object to ignite it.
To make a big fire, combustible oily substance precedes the rocket,
and after the fire is started, more oil-ladles are added to feed
and expand the fire.

In the Tang Dynasty there was another kind of fire arrow which
employed a bow to shoot out the arrows that carried fire. Details
were also recorded in Tai Pai Yin Chin, vol. 4. It goes as follows:
"Fire arrow: Those who can shoot 300 feet away attach a ladle of
fire onto their arrowhead and several hundreds of such arrows are
shot into the enemy's tents. Their supplies will be burned away.
One then takes advantage of the confusion to launch an attack".
Thus, a ladle of fire is attached to the arrowhead and used as a

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[1] Wei Kung Ping Fa edited by Li Ching is lost. Se vol. 3 of Wei
Kung Ping Fa Chi Pen, edited by Wang Tsung I (Ch'ing Dynasty).

fire starter to burn the object. This is different from the
above-mentioned rocket which first thrusts off oil ladles to aid
combustion. However, these are similar in principle in that a
fire starter is slung out via elastic force.

Since the Sung Dynasty, rockets have found more and more fre-
quent use on battlefields. "Ping Kao" vol. 131 of Hsü Wen Hsien
T'ung Kao recorded that in April of the 8th year of Tien Huey's
rule of Chin, (1130 A.D.), there was a battle between Chin and Nan
Sung. The Chin soldiers "shot rockets from the light boats. Smoke
and flame covered the whole place and the enemy was badly defeated".
The following year, in the battle of Tang T'u between Chin and
Sung, the Chin soldiers "shot rockets and burned the lookout tower"
of Sung's city wall. (Yüeh Yen Jo: Yün Lu Man Ch'ao, vol. 7).
Similar accounts abound. For example, in Hsiang Hsien Shou Ch'eng
Lu, it is recorded that in April of the 2nd year of Kai Hsi (1206
A.D.), the army defending Hsiang Hsien used "explosive arrows to
shoot and burn" the Chin soldiers. And in "Ping Kao" vol. 131 of
Hsü Wen Hsien T'ung Kao, there is an account of the "fire arrows"
used by the Mongolian army in their battle against the Nan Sung in
the 12th year of Shih Tsu of Yuan (1275 A.D.) to burn their tents,
etc. The "rocket", "explosive arrow" and "fire arrow" mentioned
in these accounts were all arrows shot out to set fire and differ
from the real rocket which is propelled by the force generated by
igniting explosives.

Nevertheless, one should realize that the ancient "rockets"
would lead to the invention of the true rockets. The ancient
"rocket" was thrust out by the elastic force in the bow and had
igniting power. After it left the bowstring, it stopped receiving
any forward push. The true rocket, however, continues to be acted
upon by the reaction to the gas jet produced by combustion during
the flight. After the invention of explosives, one would be natur-
ally led to apply explosives or fuses to the arrow. After "rockets"
had been used for a long time, one would eventually come to realize

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that after the explosive had been ignited, it sometimes would
give the arrow a sustained propulsion. When people started doing
this consciously, the rocket was born.

II. THE INVENTION OF THE ROCKET

The explosive is one of our four world renowned major inven-
tions. It is a necessary factor for the invention of the rocket.
With the equipment of explosives, "rockets" were able to evolve
into the real rocket.

Research on History of Chemistry of China reveals that the
"method to master fire" that appeared as early as in the Tang
Dynasty employed basically the three ingredients of an explosive,
viz. sodium nitrate, sulfur and carbon. Three recipes [1] for
explosives were given in Wu Ching Tsung Yao compiled by Tsen Kung
Liang of Pei Sung (1040 A.D.), showing that explosives had been
concocted in the 10th Century at the latest.

The rocket was probably invented some time between the 10th
and 11th Centuries. As recorded, in the third year of Kai Pao of
the Sung Dynasty (970 A.D.), "Defense Department Official Feng Chi
Shen et al. suggested the method of rockets. Order was given to
test it out and gifts were bestowed of them". In the third year
of Hsien P'ing (1000 A.D.), "Ch'ang T'ang Fu of the Shen Wei Shui
troop contributed the rockets, fireballs and fire caltrops that he
made". ("Ping Chih Ti I Pai Wu Shih", vol. 197 of Sung Shih) in
the above records, the rockets in question were presented to the
court as a new weapon. Therefore, these could not be the "rockets"
that existed in the Tang Dynasty. That is why they had to be
tested out so that the ruler could personally witness their power
and effect. If these were of the old type of "rockets", then they
needed not to be tested. The T'ai Tsu and T'ai Tsung brothers
of the Sung Dynasty both received military training and should have
been familiar with that type of "rocket". Hence, we have reason

to believe that the rockets separately offered by Fung Chi Cheng
and T'ang Fu were the precursors of the true rocket. In Vol. 12
of Wu Ching Tsung Yao Ch'ien Chi is recorded: "To set off the
explosives arrow, pour five ounces of explosives into the end of
the arrow to which birch bark had been added; ignite and set off".
This kind of "explosives arrow" was probably the type that Fung
and T'ang offered and probably was a variation of the true rocket.
The following premature viewpoints are presented to invite dis-
cussion. First, it was mentioned that the "explosives arrow" was
to be "set off". The words "set off" deserve special attention.
They can only apply to the true rocket. For the "rocket" of the
Tang Dynasty, the appropriate words should be "shot off" instead
of "set off". Secondly, based on the wording in "ignite and set
off", one can see that the dynamic power of the rocket came from
ignition. Obviously, the rocket was pushed off by the thrust pro-
duced by the combustion of the explosive. Thirdly, in these
accounts no mention was made of burning the opponent's tents. From
this, it is apparent that the rockets were not the "rockets" used
for setting fire. Otherwise, one should encounter such phrases as
to "burn down the enemy's tents". Besides, "pour five ounces of
explosives into the end of the arrow" was certainly not the manner
in which the "bow-slung fiery pomegranate arrow" (Figure 1) was
loaded. According to Huo Lung Ching, written in the early Ming
Dynasty, to use a "bow-slung fiery pomegranate arrow", one would
"wrap the explosive with two or three layers of tissue paper,
insert the shaft of the arrow, form into the shape of a pomegranate,
wrap tightly with burlap, seal with melted pine resin, ignite the
fuse and shoot off only after combustion has taken place". This
is different from "pouring" in the explosive. In order to pour in
the explosive, one would need a cylindrical object of sizable open-
ing to pour into.

The three points mentioned above are evidence that the "explo-
sives arrow" alluded to in Wu Ching Tsung Yao Ch'ien Chi was a true
rocket. The invention of the rocket probably took place in the

latter part of the 10th Century, i.e., early Pei Sung. From the time it was invented to the time it was included in books, there ought to be a certain time interval. The time lag between the time Fung Chi Shen offered his method of rockets and the time Wu Ching Tsung Yao was completed is 71 years, and that between the moment T'ang Fu offered his rockets and the completion of Wu Ching Tsung Yao is 40 years. It is logical that only after the value of the rocket had been borne out by actual usage in battles was the rocket included in books on military strategy.

Up until now, most people believe that the true rocket was invented by our people during Chin or Nan Sung. Based on Chou Mi's Wu Lin Chiou Shih, Liu Hsien Chou estimated the time to be around 1250 A.D. [2]. In a book recently published in Shanghai, it is said that the rocket was "invented in the latter part of Nan Sung" [3]. Somebody even said vaguely that Wu Pei Chih contains accounts of the invention of rockets by our people before the Yüan and Ming Dynasties. Europeans, e.g., O. G. Sutton and Herbert S. Zim, believe that the true rocket was employed by the Ching people in the battle being Ching and Sung in 1232 A.D. [2]. We regard these statements as inaccurate as they have set the time of invention of the rocket by our people later by more than 200 years. Japanese Chi Tien Kuang Pang, in a composition devoted to military tactics of the Sung and Yüan eras, only quoted from Sung Huey Yao the account of T'ang Fu offering the rocket and did not explain what kind of rocket it was [5].

In the past, most of the reference material used by our scholars came from Wu Pei Chih. Although its accounts of the rocket are valuable, they were given too late. This book was compiled by Mao Yüan I in the first year of Tien Hu in the late Ming Dynasty (1612 A.D.). Actually, most of the material was already available in the early Ming Dynasty.
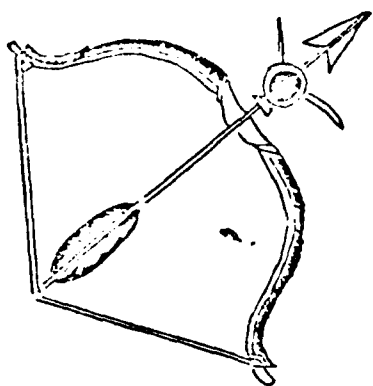
火箭全式　龙形箭架　火箭

Figure 1.   Bow-slung fiery
pomegranate arrow
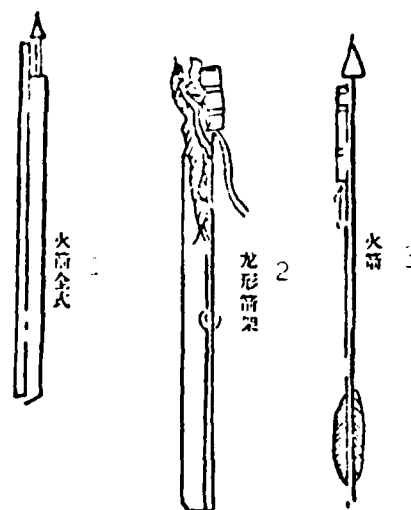(Chiao Yü: Huo Lung Ching
vol. 2)

Figure 2.   Rocket
(Chiao Yü: Huo Lung Ching vol. 2)
1--assembled rocket; 2--dragon-
shaped stand; 3--rocket

The aforementioned Huo Lung Ching was written in the 10th
year of Yung Le (1412 A.D.) which is over 200 years earlier than
Wu Pei Chih. Much of the material and many figures of the latter
were adopted from the former. The author of Huo Lung Ching,
Chiao Yü, was a specialist of weapons. The book consists of three
volumes. Not many printed copies were made and hence are hard to
find. I will write another paper specially devoted to this book.

The simplest true rocket was described in Huo Lung Ching in
words and with figures (Figure 2). It was written that one would
"use a small bamboo stake of length four feet and two inches, an
iron arrowhead 4-1/2 inches long and a four inch long iron pendant
to be attached to the end. Tie a short cylinder in the front for
making fire. When ready to set off, place in a dragon-shaped
stand or bamboo container, whichever is more convenient". Here
again, the words "set off" was used instead of "shoot off". That
the rockets had to be placed in a dragon-shaped stand" or "bamboo
container" supports our interpretation of the accounts in Wu Ching
Tsung Yao. From this written record, one can see that there already
existed a uniform specification for the structure of the rocket.

Placing the rocket in a container or on a stand helps reduce
stray force components so that the rocket can be launched along
a given direction. Moreover, the momentum produced by the com-
bustion of the explosive will not be dissipated, thus enabling
the rocket to fly farther. Rockets of such design could not have
been just created, but must have had a fairly long history of
development.

III. DEVELOPMENT OF THE ROCKET

The Yüan and Ming Dynasties saw a great deal of development
of the rocket. In late Yüan and early Ming at the latest, a
method already existed for launching off heavier objects by means
of simultaneously setting off many rockets. A good example is
found in the "mysterious fire flying duck" recorded in Huo Lung
Ching (Figure 3). This type of "flying duck" was made "with bam-
boo strips, with size of a duck that weighs about one pound.
Should be elongated in shape. Use paper pulp to seal and shape.
Fill with explosives and stuff with tissue paper. Attach head
and tail. Staple paper wings at the sides so that it looks like
a flying duck. Under each wing place two big lighting sticks
which are connected to fuses brought out through a hole in the back
of the duck. First ignite the fuses. After the duck has flown
over 100 yards and is about to drop to the ground, then the body
starts to catch fire which will cause a big fire in the whole
field". The method was thus to send a combustible duck to the
enemy's side by means of many rockets, setting fire. As far as
the recording goes, the same figures found in Huo Lung Ching were
also found in another book by Chiao Yü, Huo Lung Shen Ch'i Chen Fa
(1377 A.D.) which dated back more than 30 years earlier.

In the late Yüan and early Ming Dynasties, the rocket had
another development, i.e., many rockets were set off at the same
time. According to Ming Shih Lu Yung Le, in the 2nd year of Chien
Wen (1400 A.D.), the so-called "swarm of bees" was employed in a

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Figure 3. Mysterious fire
flying duck

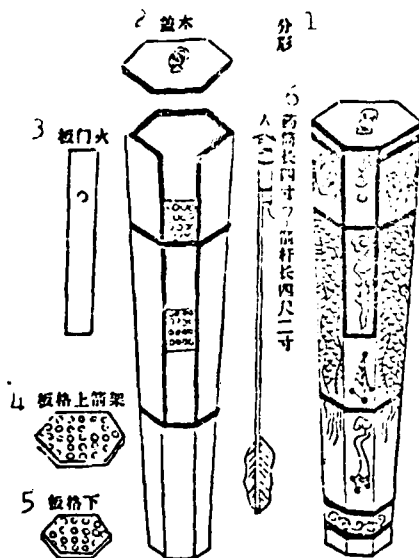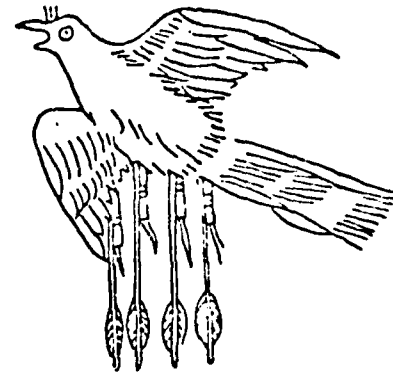(Chiao Yü: Huo Lung Ching,
vol. 3)

Figure 4. A swarm of bees
(Wu Pei Chih, vol. 127)

1--disassembled view; 2--cover;
3--door; 4--upper arrow rack;
5--lower rack; 6--4 inch
explosive cylinder; 7--4 foot,
2 inch shaft

Figure 5. Fire dragon coming out
of the water.

(Chiao Yü: Huo Lung Ching,
vol. 3)

battle. This was a powerful weapon which could "go through the
bodies of both man and horse". A "swarm of bees" was not recorded
in Huo Lung Ching, but a figure of it is found in Wu Pei Chih
(Figure 4). 32 arrows are stored in a wooden container and can be
shot out to a distance of more than 300 feet. "After the main fuse
is ignited, all the arrows are set off simultaneously". In that
book were recorded many other similar weapons, some with several
tens of arrows, others with 100 arrows, etc.

Simple two-stage rockets were already invented in the Yüan
and Ming Dynasties. A representative is the "fire dragon coming
out of the water" (Figure 5) given in vol. 3 of Huo Lung Ching.
It was recorded thus: "Take a five foot bamboo stick, remove the
nodes and scrape it thinner. Use a wooden carved dragon head in
the front and carve the end of the bamboo stick to form the dra-
gon's tail. Insert several arrows in the dragon's body and con-
nect all the fuses to an outlet in the dragon's head. When fight-
ing on water, place the dragon three or four feet above the water,
ignite and it will fly away two to three miles. From afar, it
looks just like a dragon coming out of the water. When the explo-
sives in the cylinder are almost used up, the arrows fly out to
burn all the men and the boat". One may say that such rockets are
fairly advanced. A jet propulsion mechanism was used in the spec-
ially fabricated bamboo cylinder to propel the entire flight of
the rocket. From Figure 5 one can see that four rockets were used
for the propulsion. There were also several arrows in the cylinder
which were set off during the flight. The distance of flight
reached two to three miles, showing that there was a fairly large
propelling power.

During the late Ming Dynasty, the rocket became even more ad-
vanced--it could be retrieved. In Wu Pei Chih was recorded a kind
of weapon called a "flying sand cylinder" (Figure 6). "There are
various ways of making the flying sand cylinder. If sand from the
river is not available, one can crush rocks, sift out the fine dust
with loosely woven raw silk fabric and then sift out the sand. To
each peck of sand add a liter of explosives, mix over heat and
set aside. The mouth of the firearm is made of thin bamboo slices.
There are two cylinders containing the explosives. These are to
be tied together in such a manner that the cylinder in the front
opens to the back while the cylinder in the back opens to the front.
Attach to the cylinder in the front a firecracker seven inches long
and 0.7 inches in diameter which is wrapped in three to five layers

of paper and glued to the fire starter. The prepared sand and explosive mixture is placed around the firecracker and tightly sealed. Use a small dart at the top. If the rocket is to be used on land, omit the dart. First ignite the fire starter that will cause the rocket to move forward, using a big bamboo cylinder as a guiding cylinder. Aim at the enemy's boat and the dart will hit the sail. The enemy will come out to put out the fire. At that moment, the first cylinder will explode and the sand will blind the enemy. The second cylinder will be propelled backward to our base and the enemy will never know what actually hit them". This is also a primary model of a two-stage rocket but is even better than the "mysterious fire flying duck" in that it can "be propelled back to our base".
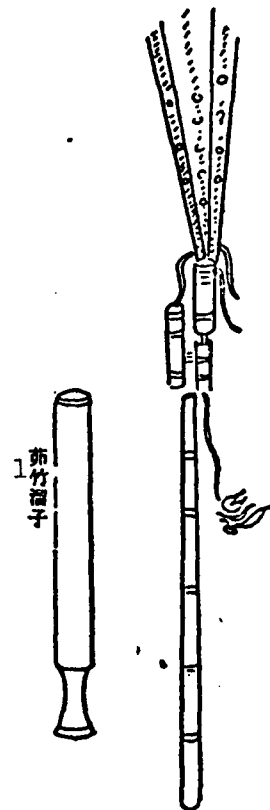
From the above, it is obvious that our working people of ancient times had many inventions in the field of the rocket and thus had many glorious accomplishments. However, in the latter part of the 18th Century, especially since 1840, our rocket technology has lagged behind. This lag is entirely due to the oppression and exploitation inflicted upon us by foreign capitalism and the reactionary government at home. In the semi-colonial, semi-feudal old China, the reactionary forces at home and abroad wildly destroyed our people's economy and scientific technology. However, even under these conditions, our working people, faced with the danger of the destruction of our nation and people, still kept on working on various weapons including the rocket in the hope that they might use these to fight the invaders. In 1894, a firearm maker in Chiang Hsi (anonymous) "made a new type of rocket,,the best of which can go five miles". However, the government at that time did not give any support to him. The test flight even invoked displeasure in the chancellor [6].

A new turn, however, finally took place. After our country was liberated in 1949, production and science had speedy development under the wise leadership of Chairman Mao and the Party. Our people

followed the instruction of the great leader Chairman Mao that
"we want to build manmade satellites, too", and did research on
rockets and satellites. On April 24, 1970, we successfully
launched our first manmade satellite of the Earth. In December
of 1975, we further accurately landed it on the Earth.

At present, our Party, headed by Chairman Hua, has summoned
us to modernize our scientific technology. As a result, our rocket
and space technology will definitely have an even greater develop-
ment, catch up with and surpass world standards, and make yet
greater contributions to mankind.

Figure 6. Flying sand cylinder
(<u>Wu Pei Chih</u>, vol. 129)
1--bamboo guiding cylinder

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